METHODS OF COMPUTING VALUES OF POLYNOMIALS

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Introduction

Recently, in the solution of problems connected with computational practice, a number of authors (N.S. Bakhvalov, A.G. Vitushkin, N.M. Korobov, S.L. Sobolev and others) have raised the question of algorithms that are not only convenient but also, in some sense, optimal. Naturally, the solution of this problem begins with the simplest cases. The computation of values of polynomials – one of the most prevailing mass operations in practical computation – gives examples of such problems and here it is possible to find optimal algorithms. In this case, in spite of the simplicity in formulating the results, it is usually necessary to use diverse and difficult methods in the construction of optimal algorithms and in the proofs that they cannot be improved. This survey is devoted to an exposition of these results and basic methods.

1. Let $P_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ be a polynomial. We are required to calculate its value at the point $x = x_0$. The simplest method consists in systematically raising x_0 to second and third etc., finally to the n-th power, then multiplying x_0^{n-k} by a_k and adding everything. Thus, n additions and $2n - 1$ multiplications are carried out.

However, there are more economical methods of computing $P_n(x_0)$. There is, for example, the well-known "Horner's scheme" by which the value of a polynomial can be computed in n multiplications and n additions. This scheme is based on the identity

$$P_n(x) = (\ldots((a_0x + a_1)x + a_2)x + \ldots + a_n).$$

We now raise the question: is it possible to improve this scheme of computation, by diminishing the number of additions or multiplications or both in comparison with Horner's method? This question demands a more exact statement of the problem. We must describe precisely what we mean by the word "scheme".

Henceforth we denote any arithmetical operation by the symbol $\circ$: addition, subtraction, multiplication or division. We denote additions and subtractions by the further symbol $\pm$ and multiplications and divisions by the symbol $\dot{\times}$. Finally, in accordance with established terminology we call the set of operations

$$p_i = R_i' \circ R_i'' \qquad (i = 1, 2, \ldots, m), \tag{0.1}$$
$$p_m \equiv P_n(x), \tag{0.2}$$

where each R_i' and R_i'' is either the variable x, or a_k, $0 \leqslant k \leqslant n$, or an absolute constant independent of x, a_0, a_1, ..., a_n or $p_j (j < i)$, leading to the identity (0.2) in x, a_0, a_1, ..., a_n a *scheme without initial conditioning of the coefficients.*

Both methods of computing $P_n(x_0)$ indicated above are obviously schemes in this sense.

In §1 of the present paper (Theorem 1.1) we show that every scheme (0.1) contains at least n operations $\pm$ and at least n operations $\dot{\times}$.

Thus, Horner's method cannot be improved for the class of *all polynomials.* Naturally this result does not exclude that there are polynomials "easier" to compute for which we shall devise "individual" computing schemes,[1] more economical than Horner's method.

For example, the computation of the polynomial $x^{15} + x^{14} + \ldots + 1 = \frac{x^{16} - 1}{x - 1} = P(x)$ requires only four operations of multiplication and one of division, i.e. 5 operations $\dot{\times}$ instead of 15 as in Horner's method:

$$p_1 = x \times x = x^2,\ p_2 = p_1 \times p_1 = x^4,\ p_3 = p_2 \times p_2 = x^8,\ p_4 = p_3 \times p_3 = x^{16},$$
$$p_5 = x - 1,\ p_6 = p_4 - 1 = x^{16} - 1,\ p_7 = p_6 : p_5 = P(x).$$

However, as will be shown in §1, for any n, the set of all "easy" polynomials of degree n is not dense in the space of all polynomials of degree not greater than n and has zero measure in it.

2. Sometimes in computational practice we repeatedly have to compute values of the same polynomial at different points (for example, the problem of computing sin x and other elementary functions). Then it is natural

[1] Absolute coefficients must not appear in "individual" schemes without initial conditioning of the coefficients, since in this case they could be functions of the coefficients of the polynomial being computed, i.e. we would have a scheme with initial conditioning of the coefficients (see below).

first to try and condition the coefficients, i.e. to manipulate (once) the coefficients of the polynomial so that the derived scheme with resulting functions of the coefficients contains as few arithmetical operations as possible.

We illustrate this idea by a very simple example due to Todd. We make use of the identity

$$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=a_0\{(x(x+\lambda_1)+\lambda_2)(x(x+\lambda_1)+x+\lambda_3)+\lambda_4\}=$$
$$=a_0x^4+a_0(2\lambda_1+1)x^3+a_0(\lambda_2+\lambda_3+\lambda_1(\lambda_1+1))x^2+$$
$$+a_0((\lambda_2+\lambda_3)\lambda_1+\lambda_2)x+a_0(\lambda_2\lambda_3+\lambda_4). \quad (0.3)$$

Equating coefficients of corresponding powers of x and solving the resulting equations, we obtain:

$$\lambda_1=\frac{a_1-a_0}{2a_0},\ \lambda_2=\frac{a_3}{a_0}-\lambda_1\frac{a_2}{a_0}+\lambda_1^2(\lambda_1+1),\ \lambda_3=\frac{a_2}{a_0}-\lambda_1(\lambda_1+1)-\lambda_2,\ \lambda_4=\frac{a_4}{a_0}-\lambda_2\lambda_3.$$

Thus, if we first compute λ_i $(i = 1, 2, 3, 4)$, then the remaining computations can be carried out by the scheme (see (0.3)):

$$\left.\begin{array}{l} p_1=x+\lambda_1,\ p_2=p_1\times x,\ p_3=p_2+\lambda_2,\ p_4=p_2+x, \\ p_5=p_4+\lambda_3,\ p_6=p_3\times p_5,\ p_7=p_6+\lambda_4,\ p_8=a_0\times p_7. \end{array}\right\} \quad (0.4)$$

The scheme (0.4) contains in all 3 multiplications (and not 4, as there would be in the standard Horner's method) and five additions. §§2, 3 and 4 will be concerned with schemes with initial conditioning of the coefficients.

The fundamental result of §2 (Theorem 2.1) establishes that every scheme with initial conditioning of the coefficients contains at least n operations $\pm$ and for $n \geqslant 2$ at least $\left[\frac{n}{2}\right]+1$ operations $\dot{\times}$ (the remarks made above about "individual" schemes and about "easy" polynomials hold here too, but it is not necessary to prohibit operations on constants).

Putting $n = 4$ we find that there must be 4 operations $\pm$ and 3 operations $\dot{\times}$ so that Todd's scheme is best possible in the multiplicative sense and almost best possible in the additive sense. §3 and §4 deal with the construction of optimal schemes of computation with initial conditioning of the coefficients. There, in particular, we have the following result (Theorem 3.2): there exists a scheme, suitable for the computation of any polynomial of degree n with complex coefficients, in which the lower bounds of Theorem 2.1 for the number of operations are attained to within one addition (for $n = 2r$):

$$\left.\begin{array}{l} p_0=1, \\ p_2=z(z+\lambda_1), \\ p_4=(p_2+\lambda_2)(p_2+z+\lambda_3)+\lambda_4, \\ p_{2s+2}=p_{2s}(p_2+\lambda_{2s+1})+\lambda_{2s+2}\quad (s=2,3,\ldots,r-1), \\ P_n(z)=\begin{cases} a_0p_{2r} & \text{for } n=2r, \\ a_0zp_{2r}+a_n & \text{for } n=2r+1. \end{cases} \end{array}\right\} \quad (0.5)$$

For polynomials of odd degree an even better scheme can be constructed

$$\left.\begin{aligned} &p_0 = z \times z = z^2,\\ &p_1 = z + \lambda_1,\\ &p_{2s+1} = p_{2s-1}(p_0 + \lambda_{2s}) + \lambda_{2s+1} \quad (s = 1, 2, \ldots, r),\\ &P_{2r+1}(z) \equiv a_0 p_{2r+1}. \end{aligned}\right\} \tag{0.6}$$

Here, in both schemes (0.5) and (0.6), the functions $\lambda_i = \lambda_i(a_0, \ldots, a_n)$ turn out, in general, to be complex for real $a_0, a_1, \ldots, a_n$. The following scheme avoids this defect:

$$\left.\begin{aligned} &p_0 = x \cdot x = x^2, \quad p_0' = p_0 + x,\\ &p_1 = x + \lambda_1,\\ &\left.\begin{aligned} &p_4^{(s)} = (p_0' + \lambda_{4s-2})(p_0 + \lambda_{4s-1}) + \lambda_{4s},\\ &p_{4s+1} = p_{4s-3} p_4^{(s)} + \lambda_{4s+1}, \end{aligned}\right\} \quad (s = 1, 2, \ldots, k),\\ &p_{4k+3} = p_{4k+1}(p_0 + \lambda_{4k+2}) + \lambda_{4k+3},\\ &P_n(x) = \sum_{l=0}^{n} a_l x^{n-l} \equiv \begin{cases} a_0 p_n & \text{for } n = 4k+1,\ 4k+3,\\ a_0 x p_{n-1} + a_n & \text{for } n = 4k+2,\ 4k+4. \end{cases} \end{aligned}\right\} \tag{0.7}$$

The identity at the end is valid for the whole domain of the variables $x, a_0, a_1, \ldots, a_n$. The functions $\lambda_i = \lambda_i(a_0, \ldots, a_n)$ are real, continuous and piecewise analytic. The stability of (0.7) for small perturbations of the coefficients follows from the continuity and the piecewise analyticity of the λ_i; all the remaining schemes in §§3 and 4 are stable in this sense.

The number of operations $\pm$ in (0.7) is $n + 1$, the number of operations $\dot{\times}$ is $\left[\frac{n+4}{2}\right]$ so that, to within one operation $\pm$ and one operation $\dot{\times}$, this scheme is optimal for even n.

3. §4 is devoted to schemes suitable for simultaneous computation of several polynomials. Such cases sometimes occur in computational practice, e.g. in the simultaneous computation of sin x and cos x. Initial conditioning of the coefficients is, of course, useful in schemes of this kind, but an extra effect of diminishing the number of operations is also possible. as one set of intermediate calculations can be used for several polynomials.

We take an example. Values of pairs of polynomials $P(x) = a_0x^2 + a_1x + a_2$ and $Q(x) = b_0x^3 + b_1x^2 + b_2x + b_3$; $a_0 \neq 0$, $b_0 \neq 0$ can be computed by the scheme

$$p_0 = x\left(x + \frac{a_1}{a_0}\right), \quad P(x) \equiv a_0 p_0 + a_2, \quad Q(x) = b_0\{(x + \lambda_1)(p_0 + \lambda_2) + \lambda_3\},$$

$$\lambda_1 = \frac{b_1}{b_0} - \frac{a_1}{a_0}, \quad \lambda_2 = \frac{b_2}{b_0} - \lambda_1 \frac{a_1}{a_0}, \quad \lambda_3 = \frac{b_3}{b_0} - \lambda_1\lambda_2,$$

where four multiplications and five additions are used (if the operations for the initial computing of the values $\lambda_1, \lambda_2, \lambda_3, \frac{a_1}{a_0}$ are not counted). If we were to compute the polynomials separately, then by Theorem (2.1) we should need at least five additions and five multiplications.

4. We give some historical notes. The ideas in this article arose in a research students' seminar run by A.G. Vitushkin and V.D. Erokhin.

Essentially, the first results were established in a paper by Ostrowski [1], where the optimality of Horner's method was proved for polynomials of degrees $n = 1, 2, 3, 4$ under the additional condition that the scheme avoids division, and also by Motzkin [2] and Todd [3], where the ideas of conditioning the coefficients (Motzkin) and economical computing schemes for $n = 4$, (Todd) were proposed.

Theorem (1.1) was proved by Belaga [4], [5] for the operations $\pm$ and by the author for the operations $\dot{\times}$.

Theorem (2.1) was proved by Belaga [4], [5].

A generalisation of Todd's construction for arbitrary n (Theorem 3.2) was proposed by Belaga. Scheme (0.6) for polynomials of odd degrees and scheme (0.7) are due to the author; the latter was anticipated in a paper by Yu.L. Ketkov, where approximately $3n/4$ multiplications were needed (and not $\left[\frac{n+4}{2}\right]$, as in scheme (0.6)). The results of §4 are all due to the author. In the present survey some schemes (e.g. the schemes in §4), not discussed either in previously published articles [6]-[9] or in the reference book [10], are published for the first time. The text contains a number of acknowledgments and references. In conclusion I wish to take the opportunity to express my gratitude to A.G. Vitushkin and V.D. Erokhin who proposed these problems, and also to thank L.A. Lyusternik for his valuable advice in writing this survey, to O.B. Lupanov and B.M. Tikhomirov for active assistance in preparing the text and improving the presentation of §3, and the introduction and §1.

§1. Lower bounds for the number of operations in schemes without initial conditioning of the coefficients

THEOREM 1.1. *Any scheme without initial conditioning of coefficients has at least n operations $\dot{\times}$ and at least n operations $\pm$.*

1) We shall prove the theorem for the operations $\dot{\times}$. The proof for the operations $\pm$ can be established similarly. The result for $\pm$ was obtained by Belaga for schemes more general then (0.1).

2) We make some preliminary remarks.

We denote by E_0 the $(n + 1)$-dimensional linear space of coefficients $L(a_0, \ldots, a_n) = \alpha$. The non-zero linear functionals $F(\alpha)$ on E_0 are called *additive parameters*. The sets $E_q(t) = E_q(F_i + R_i) = E_q(F_1 + R_1, \ldots, F_q + R_q)$, given by the q equations

$$F_i(\alpha)+R_i(t)=0 \quad (i=1, 2, \ldots, q),$$

where the $F_i(\alpha)$ $(i = 1, \ldots, q)$ are additive parameters and the $R_i(t)$ $(t = 1, 2, \ldots, q)$ are rational functions of t with numerical coefficients, are called *parametric sets*. In the definition of parametric set we use the parameter t, which must not be confused with additive parameters. We say that the additive parameter $F(\alpha)$ is a constant on the parametric set $E_q(t) = E_q(F_i + R_i)$ if

$$F(\alpha)=\sum_{i=1}^{q}\beta_i F_i(\alpha),$$

i.e. if $F(\alpha)$ is constant on E_q for any fixed value of t.

DEFINITION 1.1. We say that the operation $\dot{\times}$ used in the equation $p = R' \dot{\times} R''$ is *active on the parametric set* $E(t) \subseteq E_0$ if:

1) at least one of the functions R' or R'' is not a rational function of t and x on $E(t)$ with numerical coefficients;

2) the result of p applications of the given operation is not proportional to R' nor to R''.

For example: $a_k \cdot x$ or $a_k \cdot a_s$ are active on E_0; $x \cdot x = x^2$, $2a_k$ are not active on E_0; in general, as is easily seen, the common form of a rational function of α and x on the parametric set $E(t) \subseteq E_0$, not containing an operation $\dot{\times}$ active on $E(t)$, obtained from a scheme of the form (0.1) is $F(\alpha) + R(x;\ t)$, where $F(\alpha)$ is an additive parameter or zero and $R(x;\ t)$ is a rational function of x and t with coefficients independent of α.

NOTE 1.1. If $E_2 \subset E_1$ and the operation $\dot{\times}$ is active on E_2, then it is active on E_1. The converse is not always true.

We pass on to the proof of the theorem. Put $t = x$.

Let $p_{l_1} = p_{l_1}(\alpha,\ x) = R'_{l_1} \times R''_{l_1}$ be the first operation $\dot{\times}$ in the scheme (0.1) active on E_0.

It follows from what was said above that

$$p_{l_1}(\alpha,\ x)=R'_{l_1}\dot{\times} R''_{l_1}=(F'+R'(x))\dot{\times}(F''+R''(x))=$$
$$=(F(R'_{l_1})+R'(x))\dot{\times}(F(R'_{l_1})),\ +R''(x)),$$

where $R'(x)$, $R''(x)$ are independent of α on E_0, each of $F' = F(R'_{l_1})$ and $F'' = F(R''_{l_1})$ is an additive parameter or constant on E_0, and where at least one of F', F'' is not a constant.

If F'' is not constant, we define the parametric set E_1 by the relation

$$F_1(\alpha)+R_1(x)+\beta_1=0,$$

where $F_1(\alpha) = F''$, $R_1(x) = R''(x)$; we choose the constant β_1 so that no function $p_s(\alpha,\ x)$ $(s = 1,\ 2,\ \ldots,\ m)$ is identically zero on E_1 if $p_s(\alpha,\ x) \neq 0$ on E_0.

If F'' is a constant, then we proceed in a similar fashion putting $F_1(\alpha) = F'$, $R_1(x) = R'(x)$.

Hence $E_1(t) = E_1(F_1 + \beta_1 + R_1)$ is a set given by the equation

$$F_1+R_1+\beta_1=0,$$

where

$$F_1=\begin{cases} F(R''_{l_1}) \text{ if } F(R''_{l_1}) \text{ is not constant on } E_0, \\ F(R'_{l_1}) \text{ if } F(R''_{l_1}) \text{ is constant on } E_0, \end{cases}$$

$$R_1=\begin{cases} R''_{l_1} - F_1 \text{ if } F(R''_{l_1}) \text{ is not constant on } E_0, \\ R'_{l_1} - F_1 \text{ if } F(R''_{l_1}) \text{ is constant on } E_0, \end{cases}$$

and β_1 is chosen so that no function $p_s(\alpha, x)$ $(s = 1, 2, \ldots, m)$ is identically zero on E_1 if $p_s \not\equiv 0$ on E_0.

We deduce that

a_1) the l_1-th operation $\dot{\times}$ is not active on E_1, because p_{l_1} is proportional on E_1 either to R''_{l_1} or R'_{l_1} or $1/R''_{l_1}$, where in the last case, as is evident from the construction of E_1, $R''_{l_1} \equiv R_1(x)$ is independent of α;

b_1) there is a linear functional $F_1(\alpha) = F(p_{l_1})$ such that $F(p_{l_1}) - p_{l_1}$ is independent of α on E_1.

Let the l_2-th operation be the first $\dot{\times}$ in the scheme (0.1) that is active on E_1. It follows from Note (1.1) that $l_2 > l_1$.

In the same way as before we construct the set

$$E_2 = E_2(F_1 + R_1 + \beta_1 = 0,\ F_2 + R_2 + \beta_2 = 0),$$

where

$$F_2 = \begin{cases} F(R''_{l_2}) & \text{if } F(R''_{l_2}) \text{ is not constant on } E_1, \\ F(R'_{l_2}) & \text{if } F(R''_{l_2}) \text{ is constant on } E_1, \end{cases}$$

$$R_2 = \begin{cases} R''_{l_2} - F_2 & \text{if } F(R''_{l_2}) \text{ is not constant on } E_1, \\ R'_{l_2} - F_2 & \text{if } F(R''_{l_2}) \text{ is constant on } E_1, \end{cases}$$

and β_2 is chosen so that no function $p_s(\alpha, x)$ $(s = 1, 2, \ldots, m)$ is identically zero on E_2 if $p_s \not\equiv 0$ on E_0.

We deduce similarly that:

a_2) there is no operation $\dot{\times}$ active on E_2 in the first l_2 operations;

b_2) there are linear functionals $F_s(\alpha) = F(p_s)$ such that $F(p_s) - p_s$, for $s \leqslant l_2$, are equal on E_2 to rational functions of x with numerical coefficients.

The process of constructing parametric sets E_p continues until, for some $p = r$ in the scheme (0.1), there is no operation $\dot{\times}$ active on E_p.

Evidently, the number of operations $\dot{\times}$ active on E_0 is not less than r. To complete the proof we show that $r \geqslant n$.

From the properties b_s) $(s = 1, 2, \ldots, r)$ we deduce that on E_r $F(p_s) - p_s$ $(s = 1, 2, \ldots, m)$ are rational functions of x with numerical coefficients.

Consequently, on E_r, p_m can depend only on x_1, F_1, F_2, $\ldots$, F_r and $F(p_m)$, i.e. p_m depends either on r or $r \pm 1$ additive parameters, according as $F(p_m)$ is constant on $F(p_m)$ is constant on E_r or not. [1]

However, $p_m = P_n(x)$, i.e. p_m depends on $n + 1$ additive parameters. Hence $r \geqslant n$. The proof of the theorem is now complete.

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1 Thus, the polynomials being computed by (0.1) depend on not more than $k + 1$ additive parameters, where k is the number of operations $\dot{\times}$ in the given scheme. Because (0.1) contains only arithmetical operations, and a finite number at that, that dependence is rational. See [4] for analogous connections between the number of parameters on which p_m depends and the number of operations $\pm$ in (0.1).

Turning to the question about "individual" schemes of computation without initial conditioning of the coefficients (see Introduction, p.107), we note that if the number of all arithmetical operations in the schemes is bounded by some constant, then there are only a finite number of different schemes. Hence, and from the proof of Theorem 1.1 (see the footnote on the previous page) we deduce that *in the class of all polynomials of degree n (where n is any natural number), for almost every polynomial (in the sense of the measure M^{n+1}) within some neighbourhood, Horner's method is the most economical in relation to the number of operations $\dot{\times}$ and $\pm$ among all "individual" schemes without initial conditioning of the coefficients.*

If the scheme of computation is designed not for the class of all polynomials, but for some sub-class $\mathfrak{P}$, then it is natural in this scheme to allow operations on constants relative to $\mathfrak{P}$ (but not relative to polynomials outside $\mathfrak{P}$). We shall call such schemes *$\mathfrak{P}$-schemes* (0.1). The identity $P_n(x) \equiv p_n$ in them must be regarded as an identity in x and in $a_0, \ldots, a_n$ on $\mathfrak{P}$.

$\mathfrak{P}$-schemes (0.1) can contain a smaller number of arithmetical operations than Horner's method, as, for example, with $\mathfrak{P} = \{(a_0, a_0, \ldots, a_0)\}$, $n = 2^k - 1$ (see the example on page 106). In this case, however, almost all polynomials of degree n within some neighbourhood must remain outside the class $\mathfrak{P}$, since one can deduce the following result from the proof of Theorem 1.1.[1]

THEOREM 1.2. *If a $\mathfrak{P}$-scheme (0.1) contains at most $n - k$ operations $\dot{\times}$ or at most $n - k$ operations $\pm$, then the set $\mathfrak{P}$ generates in the space*

$$E_0 = \{(a_0, a_1, \ldots, a_n)\}$$

a rational surface of dimension not greater than $n + 1 - k$.

§2. Lower bounds for the number of operations in schemes with initial conditioning of the coefficients

1. Statement of the problem. We define schemes with initial conditioning of the coefficients by a chain of equations

$$\left.\begin{aligned} p_i &= R_i' \circ R_i'' \; (i = 1, 2, \ldots, m), \\ P_n(x) &\equiv p_m, \end{aligned}\right\} \tag{2.1}$$

which only differ from (0.1) of the introduction (see p. 106) by allowing operations to be carried out on any real functions of the coefficients of the polynomial to be computed (and not only on absolute constants and the coefficients $a_0, \ldots, a_n$, as was done in §1). The schemes (0.4)-(0.7) previously mentioned on pp. 107-108 are examples of such schemes.

Similarly to the $\mathfrak{P}$-schemes (0.1) (see p. 106) there correspond $\mathfrak{P}$-schemes (2.1) in which, in contradistinction to (2.1), the equation

[1] See footnote on previous page.

$P_n(x) \equiv p_m$ is satisfied identically, not by all values of x, a_0, a_1, ..., a_n, but by all x and all possible values of a_0, ..., a_n from the class $\mathfrak{P}$. Evidently the schemes (2.1) are special cases of $\mathfrak{P}$-schemes (2.1), when the class $\mathfrak{P}$ consists of all possible values of the coefficients.

All functions of the coefficients appearing in $\mathfrak{P}$-schemes (2.1) are called *parameters* and are written as

$$\lambda_k = f_k(a_0, \ldots, a_n) \quad (k = 1, 2, \ldots, r). \tag{2.2}$$

The equality $P_n(x) \equiv p_m$ in $\mathfrak{P}$-schemes (2.1) means that $P_n(x)$ can be expressed as a rational function of x and of the parameters λ_1, λ_2, ..., λ_r. We note that the $\mathfrak{P}$-schemes (0.1) are special cases of $\mathfrak{P}$-schemes (2.1),[1] when $P_n(x)$ takes the form of a function of x, a_0, a_1, ..., a_n.

The main object of this section is the derivation of lower bounds for the number of arithmetical operations in schemes with initial conditioning of the coefficients. These bounds will be obtained by establishing a dependence between the number of arithmetical operations in $\mathfrak{P}$-schemes (2.1) and the dimension of $\mathfrak{P}$.

2. Dependence between the dimension of $\mathfrak{P}$ and the set of parameters used in $\mathfrak{P}$-schemes (2.1). There is a simple but important lemma.

LEMMA 2.1 (Belaga). *If at most r parameters are used in a $\mathfrak{P}$-scheme* (2.1), *then $\mathfrak{P}$ generates a surface of dimension at most r in the space* $\{(a_0, a_1, \ldots, a_n)\}$.

PROOF. From the condition $P_n(x) \equiv p_m(x, \lambda_1, \ldots, \lambda_r)$ we deduce that

$$a_k = \varphi_k(\lambda_1, \lambda_2, \ldots, \lambda_r) \quad (k = 0, 1, \ldots, n), \tag{2.3}$$

where all the φ_k are rational functions, since only arithmetical operations are used in a $\mathfrak{P}$-scheme (2.1) and only a finite number of them. So Lemma 2.1 is proved.

In what follows we need the following definition.

DEFINITION 2.1. We call a $\mathfrak{P}$-scheme of the form (2.1) *minimizing* $\pm$ (or *minimizing* $\dot{\times}$) for another $\mathfrak{P}$-scheme of the form (2.1) (assuming that both schemes correspond to the same class $\mathfrak{P}$) if the first scheme contains no more operations $\pm$ (or $\dot{\times}$) than the second.

For example, the scheme mentioned on p. 107 for $n = 4$ is a $\mathfrak{P}$-scheme of the form (2.1) minimizing $\dot{\times}$ for Horner's method, and for any $\mathfrak{P}$ Horner's method is a $\mathfrak{P}$-scheme of the form (2.1) minimizing $\pm$ for the schemes on p. 107.

3. Dependence between the number of additions and subtractions and the number of parameters in a $\mathfrak{P}$-scheme (2.1). Let a $\mathfrak{P}$-scheme (2.1) be given. We select in it all those rows in which $p_i = R_i' \pm R_i''$ and re-label the p_i in them in order of occurrence as: $\tilde{p}_1$, $\tilde{p}_2$, ..., $\tilde{p}_k$. It turns out that not more than one new independent parameter is actually introduced into the scheme as a result of the operations undertaken for the transition from $\tilde{p}_s$ to $\tilde{p}_{s+1}$. For example, the operation $\lambda_1\lambda_2 x$ in fact increases the number of parameters in the scheme by only 1, since we can use

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1 If the class $\mathfrak{P}$ consists of a single element, then the parameters are constant on $\mathfrak{P}$. Consequently, in this case, $\mathfrak{P}$-schemes (2.1) are also $\mathfrak{P}$-schemes (0.1).

the product $\lambda_3 x$, where $\lambda_3 = \lambda_1\lambda_2$ is a parameter, instead of the product $\lambda_1\lambda_2 x$ in later calculations. This gives us the required dependence expressed in the following lemma.

LEMMA 2.2 (Belaga). *For each* $\mathfrak{P}$*-scheme* (2.1) *using not more than* k *operations* $\pm$, *a minimizing* $\mathfrak{P}$*-scheme of the form* (2.1), *involving at most* $k+1$ *parameters, can be constructed.*

A rigorous proof of Lemma 2.2 is given in [4]; it is similar to the proof of Lemma 2.3 (see below).

4. Dependence between the number of multiplications and divisions and the number of parameters in $\mathfrak{P}$-schemes (2.1). Let a $\mathfrak{P}$-scheme (2.1) be given. We select in it all those rows in which $p_j = R_j' \dot{\times} R_j''$ and re-label the p_j in them in order of occurrence as $\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_l$. Just as in the deduction of a lower bound for the number of operations $\pm$ we find that $\overline{p}_{s+1}$ actually contains not more than two new independent parameters in comparison with one of $\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_s$. For example, the operation $x+\lambda_1+\lambda_2$ gives only one new independent parameter $\lambda_3 = \lambda_1+\lambda_2$. To obtain a formal proof we examine a reduced expression of the $\mathfrak{P}$-scheme (2.1):

$$\left.\begin{aligned} \overline{p}_j &= T_j' \dot{\times} T_j'' \quad (j=1, 2, \ldots, l),\\ p_m &= T_{l+1}', \end{aligned}\right\} \tag{2.4}$$

where

1) $T_j' = U_j' + Z_j'$, $T_j'' = U_j'' + Z_j''$, $j = 1, 2, \ldots, l+1$;

2) each U_j', U_j'' is a linear combination with integer coefficients of the parameters $\lambda_1, \ldots, \lambda_r$;

3) each Z_j', Z_j'' is a linear combination with integer coefficients of x and all $\overline{p}_i$, where $i < j$.

Clearly, both schemes: the $\mathfrak{P}$-scheme (2.1) and (2.4) – contain l operations $\dot{\times}$. We define:

$$U_{l+1}' = \overline{\lambda}_{2l+1}, \quad U_j' = \overline{\lambda}_{2j-1}, \quad U_j'' = \overline{\lambda}_{2j} \quad (j=1, 2, \ldots, l). \tag{2.5}$$

We replace each U_j', U_j'' in (2.4) by the appropriate $\overline{\lambda}_s$. We shall take $\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_{2l+1}$ obtained here as parameters in 2.5. So the required scheme has been constructed and we here obtain the following result.

LEMMA 2.3 (Belaga). *For any* $\mathfrak{P}$*-scheme using not more than* l *operations* $\pm$, *a minimizing* $\dot{\times}$ $\mathfrak{P}$*-scheme of the form* (2.1) *involving at most* $2l+1$ *parameters, can be constructed.*

5. Improvement of the estimate in 4. Let $n \geqslant 2$ and a_0 be a non-zero constant on $\mathfrak{P}$. Then, for (2.4) and hence for $\mathfrak{P}$-schemes (2.1), we can construct minimizing $\dot{\times}$ $\mathfrak{P}$-schemes in which some of the operations $\dot{\times}$ do not introduce more than one parameter each. Let us attach a more precise meaning to these words. Let a $\mathfrak{P}$-scheme have the form (2.4). We compare the j-th operation in this scheme, $j = 1, 2, \ldots, l$, with the dimension D_j of the minimal space spanned by the parameters U_q', U_q'' ($q = 1, \ldots, j$). We say that the j-th *operation* $\dot{\times}$ *introduces into this scheme* ν_i parameters if $D_j - D_{j-1} = \nu_j$. Clearly, $0 \leqslant \nu_j \leqslant 2$. We show that $\sum_{j=1}^{l} \nu_j \leqslant 2l - 1$. To begin with we assume that there are no divisions in (2.1), i.e. all the $\overline{p}_i$ are polynomials in x and $\overline{\lambda}_s$. Let $\overline{p}_j$ be the first of these with a non-integer leading coefficient, say:

$$\overline{p}_j = \overline{\lambda}_{2j-1} \times T_j'' = \overline{\lambda}_{2j-1}(Z_j'' + \overline{\lambda}_{2j}) = \overline{\lambda}_{2j-1} Z_j'' + \overline{\lambda}_{2j-1}\overline{\lambda}_{2j}.$$

If in (2.4) we put $\overline{p}_j = \overline{\lambda}_{2j-1} Z_j''$ instead of $\overline{p}_j = \overline{\lambda}_{2j-1} \times T_j''$, then we obtain a minimizing $\dot{\times}$ $\mathfrak{P}$-scheme of the form (2.1) for (2.4), containing only $2l$ parameters. This method is not always valid: $p_1 = x(x + \lambda)$, $p_2 = x^2$, $p_3 = p_2 - p_1$.

However, there is an operation $\dot{\times}$ in the scheme introducing less than two parameters, because at some point in the scheme a $\overline{p}_j$ emerges which is non-linear in x. By means of the identity

$$(a_0 x + \overline{\lambda}_1)(x + \overline{\lambda}_2) + \overline{\lambda}_3 = a_0 x\left(x + \left(\frac{\overline{\lambda}_1}{a_0} + \overline{\lambda}_2\right)\right) + \overline{\lambda}_1\overline{\lambda}_2 + \overline{\lambda}_3$$

it is not difficult to construct, for a given $\mathfrak{P}$-scheme (2.1), a minimizing $\mathfrak{P}$-scheme in which, after its reduction to the form (2.4), the inequality

$$\sum_{j=1}^{l} \nu_j \leqslant 2l - 1$$

holds. In the general case, when divisions can be performed in (2.1), the same result is obtained. To derive it we use the fact that for arbitrary rational fractions

$$R' = \frac{P'(x, \lambda)}{Q'(x, \lambda)}, \quad R'' = \frac{P''(x, \lambda)}{Q''(x, \lambda)}$$

the denominator of the product $(R' + \lambda_1)(R'' + \lambda_2)$, where λ_1 and λ_2 are parameters independent of x and λ, is equal to $Q'(\dot{x}, \lambda) \cdot Q''(x, \lambda)$ after all possible cancellations, i.e. if the operation $\dot{\times}$ introduces two parameters, cancellation is impossible. On the other hand, the final expression p_m does not contain a denominator depending on x; this can be shown, for example, by differentiating p_m $n+1$ times with respect to x.

The result of Lemma 2.3 can now be improved to the following form.

LEMMA 2.4. *If $n \geqslant 2$ and a_0 is a non-zero constant on $\mathfrak{P}$, then for any $\mathfrak{P}$-scheme of the form (2.1) using no more than l operations a minimizing $\mathfrak{P}$-scheme of the form (2.1) involving at most $2l$ parameters can be constructed.*

In the following, unless otherwise stated, we shall consider only those classes $\mathfrak{P}$ that consist of polynomials of degrees at least 2 and we shall assume that the leading coefficient a_0 is not identically zero on $\mathfrak{P}$.

6. Bound for the number of arithmetical operations. From Lemmas 2.1, 2.2 and 2.4 we deduce the following result.

THEOREM 2.1 (Belaga).[1] *If the* $\mathfrak{P}$*-scheme* (2.1), *for* $n \geqslant 2$, *makes use of at most* $n - t$ *operations* $\pm$ *or at most* $\left[\frac{n}{2}\right]+1-t$ *operations* $\dot{\times}$,[2] *then* $\mathfrak{P}$ *is a rational surface of dimension at most* $n + 1 - t$ *in the space* $\{(a_0, a_1, \ldots, a_n)\}$.

In particular, taking $\mathfrak{P} = \{(a_0, a_1, \ldots, a_n)\}$, we deduce from Theorem 2.1:

COROLLARY 2.1. *Any scheme* (2.1), *for* $n \geqslant 2$, *uses at least* n *operations* $\pm$ *and at least* $\left[\frac{n}{2}\right]+1$ *operations* $\dot{\times}$.

We denote by $\mathfrak{P}_{N,t,n}$ the union of all those sets $\mathfrak{P}$ for which $\mathfrak{P}$-schemes exist containing at most N arithmetical operations, of which either at most $n - t$ are operations $\pm$ or at most $\left[\frac{n}{2}\right]+1-t$ are operations $\dot{\times}$.

Since there is only a finite number of different $\mathfrak{P}$-schemes of the form (2.1) in which the number of arithmetical operations is uniformly bounded, we deduce from Theorem 2.1:

COROLLARY 2.2. *The set* $\mathfrak{P}_{N,t,n}$ *consists of a finite number of rational surfaces of dimension at most* $n + 1 - t$ *in the space* $\{(a_0, \ldots, a_n)\}$.

It follows from Corollary 2.2 that *for almost all (in the sense of the* M^{n+1} *measure) polynomials of degree n, within some neighbourhood, any computing scheme independent of x and using only arithmetical operations contains at least* $\left[\frac{n}{2}\right]+1$ *operations* $\dot{\times}$ *and at least n operations* $\pm$.

NOTE 2.1. All the results and proofs of §§1 and 2 can be generalized without any essential changes to the case when the coefficients, variables, and constants are complex.

§3. Construction of schemes with initial conditioning of the coefficients for the computation of one polynomial

In this section we shall construct schemes with initial conditioning of the coefficients in which, to within one or two operations, the bounds

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[1] In Theorem 2.1, Belaga's bound for the number of operations $\dot{\times}$ is improved by 1 (to derive Belaga's bound it is sufficient to use Lemma 2.3 instead of Lemma 2.4).

[2] As an example of $\mathfrak{P}$-schemes (2.1) using fewer than n operations $\pm$ and less than $\left[\frac{n}{2}\right]+1$ operations $\dot{\times}$ we can take the following scheme:

$$
\begin{aligned}
g_2^{(0)} &= x(x+\lambda_1),\\
g_4^{(0)} &= (g_2^{(0)}+\lambda_2)(g_2^{(0)}+x+\lambda_3),\\
p_2 &= g_2^{(0)}+\lambda_4,\\
p_{4s+2} &= p_{4s-2}(g_4^{(0)}+\lambda_{2s+3})+\lambda_{2s+4} \qquad (s=1, 2, \ldots, k-1),\\
P_n(x) &= a_0 p_{4k-2}.
\end{aligned}
$$

It is easily seen that the dimension of the set in this case is less than $n+1$.

in the preceding section are attained.

First of all we construct a scheme in which an arbitrary polynomial of degree n with real coefficients can be computed by $\left[\frac{n+4}{2}\right]$ multiplications and $n+1$ additions, using only real numbers. This is the main result of §3.

1. Lemmas about a pipe and wires and a property of the roots of polynomials with real coefficients. Our object at this point is the derivation of a property of polynomials with real coefficients, expressed in the following lemma.

LEMMA 3.1. *For any set of real numbers $d_1, d_2, \ldots, d_{n-1}$ there exist a constant $N > 0$ and a continuous piecewise-linear function $u(t)$, $-\infty < t < +\infty$, $u'(t) =$ constant < 0 for $|t| < N$, such that the polynomial in z*

$$P_n(z, t) = \sum_{l=1}^{n} d_l z^l - u(t),$$

where $n = 2k+1$ and $d_n = 1$, can be expressed in the form

$$P_n(z, t) = \prod_{l=0}^{2k} (z - z_l(t)), \tag{3.1}$$

where the $z_l(t)$ $(l = 0, 1, 2, \ldots, 2k)$ are continuous piecewise-algebraic complex functions of t and the functions

$$z_0(t),\ z_{2l-1}(t) + z_{2l}(t),\ z_{2l-1}(t) z_{2l}(t)\ (l = 1, 2, \ldots, k),$$

are continuous and real.

For convenience of presentation we construct an intuitive model of a polynomial and its roots in which the required properties will become evident.

Suppose that we are given a polynomial in x, of odd degree $n = 2k+1$ with real coefficients, of the form $u = P_n(x) = \sum_{m=1}^{n} d_m x^m$, $d_n = 1$. We place the graph of $u = P_n(x)$ in a vertical plane OXU with vertical axis OU and imagine that this graph lies inside a thin, curved, hollow *pipe*, infinite on both sides, with holes under the minima and above the maxima of $P_n(x)$ and also at the infinitely remote points of $P_n(x)$. We label all the openings of the pipe from left to right with the numbers from zero to $2Q+1$, $2Q \leqslant n-1 = 2k$; we label the segment of the pipe from the $(l-1)$-th to the l-th opening, $l = 1, 2, 3, \ldots, 2Q+1$, with the number l. Hereafter almost up to the end of the proof of Lemma 3.1 we shall assume that all the holes in the pipe are situated at distinct levels of u.

We construct the complex plane OXY perpendicular to the plane OXU with real axis OX and imaginary axis OY. In OXY we have for each value u on OU, n roots $z_j = z_j(u) = x_j(u) + iy_j(u)$, $j = 0, 1, \ldots, n-1$, of the equation $u = P_n(z)$, $z \in OXY$, $u \in OU$; $z_j(u)$ are continuous single-valued complex functions of u.

We consider the graphs of these functions in the space $OXYU$. We surround the graphs of each of them (representing an infinite connected branch) by an infinitesimally thin tube with holes at the points where $u = \pm\infty$ and nowhere else, and we regard these tubes as not communicating with one another. The given tubes must not interfere with the previously constructed pipe surrounding the graph of $u = P_n(x)$ in the plane OXU.

We remove from the graphs of the roots and from the tubes all their segments of positive length in the plane OXY, i.e. inside the pipe. The remaining parts of the tubes have holes at the extremal points of $u = P_n(x)$, where the pipe has openings. The boundaries of these holes in the pipe must coincide with the boundaries of the holes in the tubes. We do this in such a way that:

a) each segment of the pipe has continuations in the form of thin tubes running to $u = +\infty$, $u = -\infty$;

b) all segments of the pipe have a common internal region with their continuations and with each other;

c) the interior in the system "pipe-tubes" communicates with the exterior only through infinitely remote holes.

We shall regard all tubes isolated from the pipe as continuations of a null (non-existent) segment of the pipe. For $l = 0, 1, 2, \ldots, 2q + 1$, $q = Q$, we call the l-th segment together with its continuations the *l-th section of motion*. As $u \to \pm\infty$, we have one pipe and $2k$ tubes surrounding the $2k + 1$ branches of the graphs of the roots $z_j(u)$. At $u = +\infty$ the pipe and all the $2k$ tubes each have one hole and similarly at $u = -\infty$. We place in each of these holes at $u = +\infty$ the end of an infinite thin wire (different wires to different holes) and thread all the wires through the interior of the tubes (and pipe) taking the wire all the time through the points of the graphs of the roots $z_j(u)$ (fig. 1; in the figure the space $OXYU$ is projected on the plane OXU; the arrows indicate the paths of

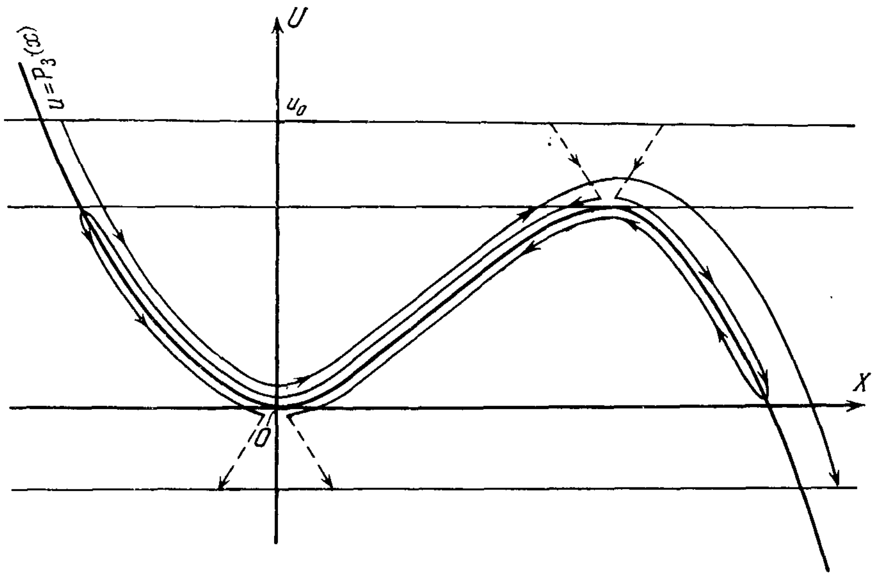

Fig. 1.

the ends of the wires: the continuous lines denote the paths inside the pipe, the dotted lines denote the paths outside the pipe and the plane OXU all drawn projected on this plane OXU). We assume that the ends of the wires are inserted into the holes at the time $t = -\infty$ and afterwards move alternately downwards and upwards, not leaving the corresponding interior of the isolated tubes or the interior of the pipe and tubes - continuations of the pipe. At any time t the ends of all the wires must have a common projection $u(t)$ on the axis OU. The following conditions must be satisfied for the progress of the ends of the wires in the interior of the pipe and tubes.

1) At any time t in each non-null section the end of each wire hangs at the same level;[1] in the null section $2k - 2q$ ends of wires hang at the same level; thus, each wire lies in some section - its *position section at the time* t.

2) All the wires must be labelled with the numbers $0, 1, 2, \ldots, 2k$ so that at each moment of time t the end of the zero wire lies inside the pipe and the ends of the $(2l - 1)$-th and $(2l)$-th wires, $l = 1, 2, \ldots, k$, either both lie inside the pipe or project on to complex conjugate points in OXY (the $(2l - 1)$-th and $(2l)$-th wires are called *pairs of wires*).

3) In a sufficiently small neighbourhood of any time t_0 either the ends of all the wires always remain in the same sections or the ends of all the wires except two, and these two wires, moving inside the pipe, exchange their position sections at the time t_0 (these sections must be neighbouring and non-null). In the latter case, and only then, the direction of the motion of $u(t)$ up and down the OU axis changes its sense at t_0.

If we put $|u'(t)| = v =$ constant, i.e. if we fix the absolute size of the rate of change of $u(t)$, then the movement of the ends of the wires can always be continued uniquely (and, moreover, for both t increasing and t decreasing) from any time t_0 for which conditions 1) and 2) are maintained and the ends of the $2k + 1$ wires are at some common level $u(t_0)$. Three cases are logically possible: either our process re-cycles, or $u(t) \to \infty$ as $t \to \infty$, i.e. the ends of all wires rise, as $t \to \infty$, to the upper infinitely remote hole in the tube (compare below with the first condition for ending the motion), or $u(t) \to -\infty$ as $t \to \infty$, i.e. the ends of the wires descend to the lowest hole of the tube (compare below with the second condition for ending the motion).

For the proof of Lemma 3.1 it is sufficient to prove that in fact only the third case can arise: $u(t) \to -\infty$ as $t \to +\infty$. To begin with we prove that re-cycling is impossible i.e. that $|u(t)| \to \infty$ as $t \to +\infty$.

Evidently properties 1)-3) can always be satisfied in the section of change of t, when t is so small that $u(t)$ lies above the projection of any extremum of the graph of $u = P_n(x)$ on OU. We adopt a numbering of the wires for which property 2) holds in this section of change of t. We change to more formal language for the construction of the proof.

---

[1] At the junction of two sections at the level $u(t)$, the ends of two of the wires hang level at the time t.

We call the totality of the following given objects the *state* A:

a) the *direction sign of the state*: plus or minus;

b) the set of *numbers of the state* is the set of integers $i_0(A)$, $i_1(A)$, ..., $i_{n-1}(A)$ from zero to $2q+1$ inclusive, $0 \leqslant q \leqslant k$, where there are $n-1-2q$ zeros among these numbers and the rest are all distinct.

As is clear from the definition, there is a finite number less than $2n!$ of different states.

We label all the consecutive times t_ν, $\nu = 1, 2, \ldots$, in which the direction of movement of the level of $u = u(t)$ along OU changes, also including here $t = \pm\infty$. We call the state A *consistent with the state of the wires in the interval of time* $(t_\nu, t_{\nu+1})$, $\nu \geqslant 0$, if:

a) the direction sign of the state A agrees with the sign of the number $u'(t)$ for values of t in the interval $(t_\nu, t_{\nu+1})$;

b) each number $i_s(A)$ of the state A, s 0, 1, ..., $n-1$, is equal to the number of the situation section of the end of the s-th wire for $t \in (t_\nu, t_{\nu+1})$.

In what follows we call the state A consistent with the state of the wires in the interval $(t_\nu, t_{\nu+1})$ briefly the *state with index* ν.

We say that a *state* B *follows* A if A has index ν and B has index $\nu+1$, $\nu \geqslant 0$.

If B follows A, then we describe B as *succeeding* A, A as *preceding* B.

If we are given a state A with index ν and a value $u = u(t)$ of a level on OU for some $t \in (t_\nu, t_{\nu+1})$, $t_{\nu+1} \neq +\infty$, then by virtue of the continuity and unique definition of movement of the level $u = u(t)$ and of the ends of the wires at any finite time, the state B with index $\nu+1$ exists and is uniquely determined.

Two states A with index ν and B with index $\nu+1$ always have opposite direction signs and identical sets of numbers, except for two numbers from each of the states A and B which change places. The numbers are positive and differ by one from one another: $i_s(A) = i_s(B)$, $s \neq p, q$, $1 \leqslant i_p(A) = i_q(A) - 1 = i_q(B) = i_p(B) - 1$.

From the above argument we deduce three simple *properties of successions of states.*

1°. If a state B has the indices ν and μ, $\nu \neq \mu$ at the same time, then a state C with index $\nu+1$ has at the same time another index $\mu+1$ and a state with index $\nu-1$ has at the same time the index $\mu-1$ (*property that the order of succession of states is independent of the time.*

2°. A state with index ν has no predecessor if and only if $\nu = 0$.

3°. A state with index ν has no successor if and only if $t_{\nu+1} = +\infty$; in this case, for some $t < t_{\nu+1}$ one of two conditions for ending the motion is satisfied:

1) $u(t)$ lies above the projection on OU of every hole in the pipe, except for the one infinitely remote, and $u'(t) > 0$; also $u(t_{\nu+1}) = +\infty$.

2) $u(\tilde{t})$ lies below the projection on OU of each hole in the pipe, except for one of the infinitely remote holes, and $u'(\tilde{t}) < 0$; in this case $u(t_{\nu+1}) = -\infty$.

LEMMA 3.2. (about the acyclicity of states). *One and the same state cannot have two distinct indices.*

PROOF. Suppose that a state A with index ν also has index $\nu + r$ where $r \geqslant 1$. Then by property 1) of the succession of states each state with index $\nu - \mu$, where $\mu = 0, 1, 2, \ldots, \nu$, has index $\nu + r - \mu$. In particular, a state with index 0 has index r, $r \geqslant 1$, and consequently has a preceding state with index $r - 1$. This contradicts property 2) of the succession of states and proves Lemma 3.2.

By the finiteness of the set of all states Lemma 3.2 leads to

COROLLARY 3.1. *There is an instant at which one of the conditions for ending the motion is satisfied.*

We shall show that the first condition for ending motion cannot hold. To do this we study in greater detail the behaviour, taken separately, of a pair of wires for all t. We attach a motion sign and a state at a given time to each pair of wires. The direction of movement of the end of any wire inside the pipe from left to right will be regarded as positive and from right to left as negative. We take the movement of a pair of wires inside the pipe as positive if the direction signs of the movement of the ends of the pair of wires are identical and negative if these signs are opposite. We regard the movement of each pair of wires outside the pipe as negative.

We define the state of the pair at time t for any given pair of the $(2l - 1)$-th and $2l$-th wires, $1 \leqslant l \leqslant k$, as the collection:

a) of a plus or minus sign, corresponding to the motion sign of the given pair at time t;

b) of two integers of the state of the pair corresponding to the numbers $i_{2l-1}(A)$, $i_{2l}(A)$ (see above, p. 120), where A is the state of the wires in the interval $(t_\nu, t_{\nu+1})$ containing t.

As the time t moves from $-\infty$ to $+\infty$, the state of a given pair, the $(2l - 1)$-th and $2l$-th wires, can change only at certain discrete critical times at which the direction of movement of $u(t)$ along OU changes. We shall ignore some of them. Namely, if the states of our pair coincide up to and just after a critical time, but the direction sign of the movement of $u(t)$ along OU is changed, then we ignore this time and afterwards we investigate the remaining critical times. After all the discards we obtain a chain of critical times $t_{\nu_1} < t_{\nu_2} < \ldots < t_{\nu_r}$, $r \geqslant 0$. To this chain we add $t_{\nu_0} = -\infty$ and $t_{\nu_{r+1}} = +\infty$.

We consider the movement signs of a given pair and the direction signs of the motion of the level of $u(t)$ along OU in sufficiently small half-neighbourhoods of the times t_{ν_j} $(j = 0, 1, \ldots, r + 1)$, not containing t_{ν_j} itself. We deduce:

a) in the transition from any time in a left half-neighbourhood of t_{ν_j} $(j = 1, 2, \ldots, r)$ to any time in a right half-neighbourhood of t_{ν_j} both signs (of movement of the pair and direction of motion of $u(t)$) change to their opposites; [1]

b) in the transition from any time in a right half-neighbourhood of

[1] In passing through a critical time the direction sign of the motion of one wire in the pair changes to its opposite and does not change for the other wire; one of the numbers $i_{2l}(A)$, $i_{2l-1}(A)$ corresponding to the first wire does not change, the other changes by one.

t_{ν_j} ($j = 0, 1, 2, \ldots, r$) to any time in a left half-neighbourhood of $t_{\nu_{j+1}}$ both signs are invariant.

Just as the motion of any pair of equal sign both in any neighbourhood of the initial time $t_{\nu_0} = -\infty$ and in any neighbourhood of the final time $t_{\nu_{r+1}} = +\infty$ turns out to be negative, so the direction sign of the motion of $u(t)$ along OU in these neighbourhoods is likewise negative. Consequently the first condition for ending motion cannot be satisfied. Hence the assertion of the main Lemma 3.1 follows. So far we have proved this for the case when all the extrema of the polynomial $P_n(x) = \sum_{l=1}^{n} d_l x^l$ lie at different levels. In particular, the polynomials $P_n(x, C) = P_n(x) + \frac{x}{C}$ have this property for any fixed set of real coefficients of the polynomial $P_n(x)$ if C is taken sufficiently large. Taking the limit as $C \to +\infty$ we deduce Lemma 3.1 in the general case.

2. Further properties of the roots of a polynomial with real coefficients. *A pair of roots of a polynomial* with real coefficients are any two of its roots which are either complex conjugates or are both real.

LEMMA 3.3. *For arbitrary real numbers M, $d_1, d_2, \ldots, d_{4j}$, $j \geqslant 1$, a real number d_0 can always be chosen such that among the roots of the polynomial in z*

$$\overline{P}_{4j+1}(z, d_0) = \sum_{l=0}^{4j+1} d_l z^l,$$

where $d_{4j+1} = 1$, there are four whose sum is real and which split up into two pairs of roots.

PROOF. We take functions $u(t)$ and $z_l(t)$ ($l = 0, 1, \ldots, 4j$) satisfying the conditions of Lemma 3.1 such that the polynomial in z $P_{4j+1}(z, t) = \overline{P}_{4j+1}(z, -u(t))$ is represented in the form (3.1). Because $u(t) \to \pm\infty$ as $t \to \pm\infty$, the roots of the polynomial $\overline{P}_{4j+1}(z, -u(t))$ are asymptotically equal (as $t \to \pm\infty$) to the roots of the equation $z^{4j+1} - u(t) = 0$ (figs. 2 and 3).

From the functions $z_l(t)$, $l = 1, 2, \ldots, 4j$ (see (3.1)) we choose two pairs: $z_{2s-1}(t)$ and $z_{2s}(t)$, $z_{2r-1}(t)$ and $z_{2r}(t)$ each of which, as $t \to +\infty$ or as $t \to -\infty$, is

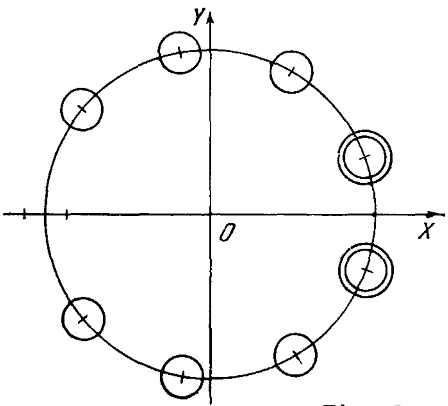

Fig. 2.

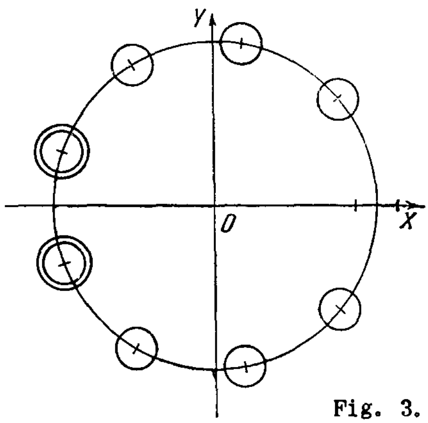

Fig. 3.

asymptotically equal to the pair of roots of $z^n - u(t) = 0$ having the maximal absolute value of the real part of all the complex roots of $z^n - u(t) = 0$ (the first pair as $t \to +\infty$ and the second as $t \to -\infty$). (In figs. 2 and 3 the domains in which these pairs lie are bounded by the double circles). If these two pairs of functions coincide, i.e. $s = r$, we select from the interval $1 \leqslant p \leqslant 2j$ an arbitrary integer p not equal to r.

Then at least one of the continuous real functions of t

$$w_1(t) = z_{2r-1}(t) + z_{2r}(t) + z_{2s-1}(t) + z_{2s}(t) \quad \text{for} \quad s \neq r,$$
$$w_2(t) = z_{2r-1}(t) + z_{2r}(t) + z_{2p-1}(t) + z_{2p}(t) \quad \text{for} \quad s = r$$

tends to $-\infty$ as $t \to -\infty$, and tends to $+\infty$ as $t \to +\infty$, and consequently takes all possible real values, including M. Lemma 3.3 is now proved.

LEMMA 3.4. *For any real numbers M, d_1, d_2, $\ldots$, d_{4p+2}, $p \geqslant 0$ a real number d_0 always exists such that among the roots of*

$\overline{P}_{4p+3}(z, d_0) = \sum_{m=0}^{4p+3} d_m z^m$, *where $d_{4p+3} = 1$, there is a pair of roots whose sum is equal to M.*

PROOF. For the coefficients of $\overline{P}_{4p+3}(z, 0)$ we find functions $u(t)$ and $z_l(t)$ $(l = 0, 1, \ldots, 4p+2)$ satisfying the conditions of Lemma 3.1, such that the polynomial $P_{4p+3}(z, t) = \overline{P}_{4p+3}(z, -u(t))$ can be expressed in the form (3.1). Using the asymptotic equality, as $t \to \infty$, of the zeros $z_l(t)$ $(l = 0, 1, \ldots, 4p+2)$ of $\overline{P}_{4p+3}(z, -u(t))$ to the roots of the equation $z^{4p+3} - u(t) = 0$ we deduce that, among the pairs of functions $z_{2s-1}(t)$, $z_{2s}(t)$ $(s = 1, 2, \ldots, 2p+1)$ there are exactly $p+1$ distinct pairs for which $z_{2s-1}(t) + z_{2s}(t) \to +\infty$ as $t \to +\infty$ and exactly $p+1$ distinct pairs for which $z_{2s-1}(t) + z_{2s}(t) \to -\infty$ as $t \to -\infty$. Because there are always $2p+1$ different pairs of functions $z_{2s-1}(t)$, $z_{2s}(t)$, at least one of them satisfies both conditions simultaneously. By the continuity of the functions $z_l(t)$ $(l = 1, 2, \ldots, 4p+2)$ we now obtain Lemma 3.4.

3. A scheme for computing polynomials of arbitrary degree with real coefficients. We shall now construct for any natural number n a computing scheme for polynomials of degree n with real coefficients with $\left[\frac{n+4}{2}\right]$ multiplications and $n+1$ additions.

To begin with we state as a lemma the following fairly evident result (see scheme (0.4) in the introduction).

LEMMA 3.5. *Let $g_4(x) = x^4 + x^3 + \beta x^2 + \beta' x + \beta''$ be a given polynomial. Then there are polynomials with numerical real coefficients, $\lambda = \lambda(\beta, \beta', \beta'')$, $\lambda' = \lambda'(\beta, \beta', \beta'')$, $\lambda'' = \lambda''(\beta, \beta', \beta'')$ in β, β', β'' such that the following identity in x, β, β', β'' is satisfied:*

$$g_4(x) = (x^2 + \lambda)(x^2 + x + \lambda') + \lambda''.$$

We now construct a computing scheme for polynomials with real coefficients (this scheme has already been mentioned in the introduction on p. 107):

$$\left.\begin{aligned}
&g_2 = x\cdot x = x^2,\\
&h_2 = g_2 + x = x^2 + x,\\
&p_1 = x + \lambda_1,\\
&\left.\begin{aligned}&g_4^{(s)} = (g_2 + \lambda_{4s-1})(h_2 + \lambda_{4s-2}) + \lambda_{4s},\\ &p_{4s+1} = p_{4s-3}g_4^{(s)} + \lambda_{4s+1},\end{aligned}\right\} (s = 1, 2, \ldots, k),\\
&p_{4k+3} = p_{4k+1}(g_2 + \lambda_{4k+2}) + \lambda_{4k+3},\\
&P_n(x) = \sum_{l=0}^{n} a_l x^{n-l} \equiv \begin{cases} a_0 p_n & \text{for } n = 4k+1,\ 4k+3,\\ a_0 x p_{n-1} + a_n & \text{for } n = 4k+2,\ 4k+4.\end{cases}
\end{aligned}\right\} \tag{3.2}$$

In scheme 3.2 the identity symbol signifies identity for all sets of real values of x, a_0, a_1, $\ldots$, a_n.

THEOREM 3.1. *Real continuous piecewise-analytic functions $\lambda_i = \lambda_i(a_0, a_1, \ldots, a_n)$ always exist for which all the equations of the scheme (3.2) are satisfied.*

PROOF. We consider first the cases when $n = 4k + 1$, $4k + 2$. We determine the coefficients of $p_{4k+1}(z)$ from the last equation in (3.2). We write the expression

$$(x^2 + x + \lambda_{4s-2})(x^2 + \lambda_{4s-1}) + \lambda_{4s} \quad (s = 1, 2, \ldots, k)$$

in the form

$$x^4 + x^3 + \beta_2^{(s)}x^2 + \beta_3^{(s)} + \beta_4^{(s)}.$$

By Lemma 3.3 we have: for arbitrary real coefficients of the polynomial

$$p_{4s+1}(x) = \sum_{q=0}^{4s+1} \alpha_q^{(4s+1)} x^{4s+1-q},$$

where s is an integer, $1 \leqslant s \leqslant k$, $a_0^{(4s+1)} = 1$, real numbers $a_1^{(4s-3)}$, $a_2^{(4s-3)}$, $\ldots$, $a_{4s-3}^{(4s-3)}$, $\beta_2^{(s)}$, $\beta_3^{(s)}$, $\beta_4^{(s)}$, λ_{4s+1} can always be found such that

$$p_{4s+1}(x) = p_{4s-3}(x)\{x^4 + x^3 + \beta_2^{(s)}x^2 + \beta_3^{(s)}x + \beta_4^{(s)}\} + \lambda_{4s+1},$$

where

$$p_{4s-3}(x) = \sum_{l=0}^{4s-3} \alpha_l^{(4s-3)} x^{4s-3-l}, \qquad \alpha_0^{(4s-3)} = 1,\ 1 \leqslant s \leqslant k.$$

Using this we now establish an iterative process for obtaining the unknown parameters $a_q^{(4s+1)}$ $(q = 1, 2, \ldots, 4s + 1)$ from the known coefficients λ_{4s+1}, $\beta_2^{(s)}$, $\beta_3^{(s)}$, $\beta_4^{(s)}$, $\alpha_l^{(4s-3)}$ $(l = 1, 2, \ldots, 4s \dot{-} 3)$. We begin this process with $s = k$ and then repeat for $s = k - 1$, for $s = k - 2$ etc., down to $s = 1$ inclusive. As a result, we obtain, in particular, the set of values of the unknown parameters $\lambda_1 = a_1^{(1)}$, λ_{4s+1} $(s = 1, 2, \ldots, k)$ and also the intermediate parameters $\beta_2^{(s)}$, $\beta_3^{(s)}$, $\beta_4^{(s)}$ $(s = 1, 2, \ldots, k)$. For each $s = 1, 2, \ldots, k$, using values already established for the parameters $\beta_2^{(s)}$, $\beta_3^{(s)}$, $\beta_4^{(s)}$, we determine the values of the unknown parameters λ_{4s-2}, λ_{4s-1}, λ_{4s} from the equation

$$g_4^{(s)}=x^4+x^3+\beta_2^{(s)}x^2+\beta_3^{(s)}x+\beta_4^{(s)}=$$
$$=(g_2+\lambda_{4s-1})(h_2+\lambda_{4s-2})+\lambda_{4s}=(x^2+\lambda_{4s-1})(x^2+x+\lambda_{4s-2})+\lambda_{4s}.$$

By Lemma 3.5 this can be done. We now have the required functions $\lambda_j = \lambda_j\ (a_0, a_1, \ldots, a_n)$ satisfying scheme 3.2 in the cases $n = 4k + 1$, $4k + 2$. In the cases $n = 4k + 3$, $4k + 4$ we obtain from the last two equations of (3.2) the values of the coefficients of $p_{4k+1}(x)$ and of the parameters λ_{4k+2}, λ_{4k+3} for which these equations are satisfied. The existence of such real values follows from Lemma 3.4. Next, for the given coefficients of $p_{4k+1}(x)$ we determine the values of all the remaining unknown parameters λ_j ($j = 1, 2, \ldots, 4k + 1$) by the method used above in the case $n = 4k + 1$. Finally, when the brackets are removed and the coefficients equated in all the equations (3.2), the problem of determining the λ_j is reduced to the solution of a system of algebraic equations, i.e., all the functions $\lambda_j = \lambda_j\ (a_0, a_1, \ldots, a_n)$ are continuous piecewise-analytic and, more precisely, are super positions of a finite number of polynomials and continuous piecewise-algebraic functions. The proof of Theorem 3.1 is now complete.

NOTE 3.1. The *stability of the scheme* (3.2) follows from the fact that all the functions $\lambda_j\ (a_0, a_1, \ldots, a_n)$ are piecewise-analytic. Stability consists in that the error in computing the scheme tends to zero when the error with which the coefficients $a_0, a_1, \ldots, a_n$ are given tends to zero, and the rates of tending to zero are proportional in both cases. All schemes we shall construct later are stable.

4. Computation of polynomials with complex coefficients. Every polynomial of degree n with complex coefficients can be computed with $\left[\frac{n+3}{2}\right]$ multiplications and n or $n + 1$ additions in which complex, not necessarily real, numbers occur, i.e., for this aspect of the problem we can indicate better computing schemes than 3.2.

THEOREM 3.2. (T.C. Motzkin, J. Todd, E.G. Belaga).[1] *For any polynomial of degree $n = 2k$ a computing scheme can be produced in which the lower bounds for the number of operations in* §2: $\left[\frac{n}{2}\right]+1$ *multiplications and n additions, are attained to within one addition* (see [4] and [5]; we quoted this scheme on p. 107, scheme (0.5)).

THEOREM 3.3. *There exists a scheme, suitable for the computation of any polynomial of odd degree, in which the lower bounds for the number of operations in* §2 *are attained to within one multiplication* (see scheme (0.6) on p. 108 and Lemma 4.3).

As far as questions about the choice of optimal schemes for computing the values of a given polynomial and about the construction of algorithms for the initial conditioning of coefficients are concerned, their answers

1 A method for constructing such schemes was first pointed out by T.C. Motzkin [2], J. Todd [3] constructed examples for $n = 4$ and 6. E.G. Belaga [4], [5] proved Theorem 3.2 for any k.

can be found in [6] and [11] together with numerous examples of the application of schemes with initial conditioning of the coefficients to the approximate computation of elementary functions.

§4. Schemes with initial conditioning of the coefficients for the simultaneous computation of the values of several polynomials

In this section we examine cases in which the values of several fixed polynomials are computed together at one and the same real or complex point common to all the polynomials, moreover, the computations are repeated many times for several values of z. Such cases are met in computational practice, for example, in the approximate simultaneous computation of two, or more, elementary functions ($\sin x$ and $\cos x$) or in problems of approximate computation with successively increasing degrees of accuracy. In computing schemes of this kind it is natural to carry out together the initial conditioning of the coefficients of the given polynomials, since an additional effect is possible later on if we make use of intermediate results for the computation of several polynomials together. As a result of such "interlacing" of schemes of computing separate polynomials, we succeed in saving roughly θq operations, where q is the number of polynomials being computed, whose degrees are greater than unity, and $\sup \theta = \frac{3}{2}$. This extra effect is particularly noticeable in the computation of several polynomials of low degrees (see the example in the Introduction on p. 108). The main object of the present section is the construction of computing schemes suitable for sets of polynomials of arbitrary degrees. We note that the construction of optimal schemes in these conditions is tied up with the resolution of certain difficulties (see the proof of Lemma 4.5). As a preliminary we find lower bounds for the number of operations. Their derivation is similar to the deduction of bounds in §2 and we can deal with them without a detailed proof.

1. Lower bounds for the number of arithmetical operations. We define *scheme* (4.1) *with initial conditioning of the coefficient for the simultaneous computation of several polynomials*

$$P_{n_i}^{(i)}(x) = \sum_{k=0}^{n_i} a_k^{(i)} x^{n_i - k}$$

of degrees n_i $(i = 1, 2, \ldots, s)$ *as a chain of arithmetical operations*

$$p_l = R_l' \circ R_l'' \, (l = 1, 2, \ldots, m), \tag{4.1}$$

where p_l, R_l', R_l'' and the symbol $\circ$ have the same meaning as in scheme (2.1) (see p. 112). However, instead of the single identity $P_n(x) \equiv p_m$ in (2.1) we have s identities

$$P_{n_i}^{(i)}(x) \equiv p_{m_i},$$

where $m_i \leqslant m$ $(i = 1, 2, \ldots, s)$. These identities are satisfied for all x and for all sets of values of the coefficients of the polynomials $P_{n_i}(x)$ $(i = 1, 2, \ldots, s)$. Together with this we can consider "individual" computing schemes (4.1) in which the sets of coefficients are fixed, but the identity is satisfied only by x. In the derivation of a lower bound for the number of operations it does not matter whether the variable x and the coefficients of the polynomials $P_{n_i}^{(i)}(x)$ take only real or all complex values. To be definite we shall assume that we choose the real case at this point.

Using the technique developed in §2 it is not difficult to obtain the following generalisation of Theorem 2.1.

THEOREM 4.1. *Let $n_1, n_2, \ldots, n_s$ be a set of s natural numbers, $s \geqslant 1$, among which at least one is greater than 1. Then every scheme* (4.1) *for the simultaneous computation of the polynomials*

$$P_{n_i}^{(i)}(x) = \sum_{k=0}^{n_i} a_k^{(i)} x^{n_i-k}$$

of degrees n_i with variable and independent coefficients $a_l^{(i)}$ $(l = 0, 1, \ldots, n_i)$, $a_0^{(i)} \neq 0$ $(i = 1, 2, \ldots, s)$, contains at least $N - s$ operations of addition and subtraction and at least $\left[\frac{N-s+2}{2}\right]$ operations of multiplication and division, where

$$N = s + \sum_{i=1}^{s} n_i$$

is the total number of coefficients in all the polynomials $P_{n_i}^{(i)}(x)$ $(i = 1, 2, \ldots, s)$.

If $\mathfrak{P}_{N,s,t,M}$ is the class of all possible sets of coefficients $a_l^{(i)}$ $(l = 0, 1, \ldots, n_i;\ i = 1, 2, \ldots, s)$, $a_0^{(i)} \neq 0$, for which there exists "individual" schemes (4.1) *for the simultaneous computation of the polynomials $P_{n_i}^{(i)}(x)$ each consisting of not more than M arithmetical operations among which there are either at most $N - s - t$ additions and subtractions or at most $\left[\frac{N-s+2}{2}\right]$ multiplications and divisions (M, t are previously fixed finite natural numbers), then the $\mathfrak{P}_{N,s,t,M}$ lies in the union of a finite number of rational surfaces of dimension at most $N - t$ in the N-dimensional space of coefficients of the polynomials.*

The derivation of Theorem 4.1 hardly differs in principle from that of Theorem 2.1. We note only that in the scheme (4.1) parameters may "enter" s times "without the help" of multiplications and divisions and s times "without the help" of additions and subtractions (and not only once as in §2), and this affects the lower bound by reducing it.

We turn now to the construction of a scheme for the simultaneous computation of the values of several polynomials. If we take the computations in the schemes in §3 separately for each polynomial, then we must use approximately $\theta' s$ more operations in the computation than in the bound of Theorem 4.1, where $\sup_{n_i} \theta' = 2$. This difference between upper and lower bounds can be decreased significantly, sometimes even abolished, in schemes with combined use of intermediate results.

2. Some auxiliary results for the construction of schemes for the simultaneous computation of the values of several polynomials with complex coefficients. We write down two systems of equations

$$\left.\begin{aligned} &\alpha_l^{(s)} + \alpha_{l-1}^{(s)}\mu = \alpha_l^{(s+1)} \quad (l=1, 2, \ldots, s), \\ &\mu\alpha_s^{(s)} + \mu' = \alpha_{s+1}^{(s+1)}, \end{aligned}\right\} \tag{4.2}$$

where

$$\alpha_0^{(s)} = 1,$$

$$\left.\begin{aligned} &\alpha_l^{(s)} + \alpha_{l-1}^{(s)}\lambda + \alpha_{l-2}^{(s)}\lambda' = \alpha_l^{(s+2)} \quad (l=1, 2, \ldots, s+1), \\ &\alpha_s^{(s)}\lambda' + \lambda'' = \alpha_{s+2}^{(s+2)}, \end{aligned}\right\} \tag{4.3}$$

and where

$$\alpha_{-1}^{(s)} = \alpha_{s+1}^{(s)} = 0, \quad \alpha_0^{(s)} = 1.$$

Then we have the obvious lemma:

LEMMA 4.1. *Let*

$$p_r(z) = \sum_{j=0}^{r} \alpha_j^{(r)} z^{r-j} \qquad (\alpha_0^{(r)} = 1)$$

($r = s$, $s+1$, $s+2$, s is a natural number) be polynomials with complex or real coefficients and arbitrary complex or real numbers μ, μ', λ, λ', λ''. Then the identity in z

$$p_{s+1}(z) \equiv p_s(z)(z+\mu) + \mu'$$

is equivalent to the system of equations (4.2), and the identity in z

$$p_{s+2}(z) \equiv p_s(z)(z^2 + \lambda z + \lambda') + \lambda''$$

is equivalent to the system of equations (4.3).

LEMMA 4.2. *Let*

$$p_r(z) = \sum_{j=0}^{r} \alpha_j^{(r)} z^{r-j}, \qquad (\alpha_0^{(r)} = 1)$$

($r = s$, $s+2$, s is a natural number) be polynomials with complex or real coefficients. Then the identity in z

$$p_{s+2}(z) \equiv p_s(z)(z^2 + \lambda') + \lambda''$$

is equivalent to the following systems of equations

$$\left.\begin{aligned} &\alpha_l^{(s)} = \alpha_l^{(s+2)} - \alpha_{l-2}^{(s)}\lambda' \quad (l=1, 2, \ldots, s), \\ &\alpha_s^{(s)}\lambda' + \lambda'' = \alpha_{s+2}^{(s+2)}, \\ &\sum_{j=0}^{\nu} \alpha_{s+1-2j}^{(s+2)}(-\lambda')^j = 0, \end{aligned}\right\} \tag{4.4}$$

where

$$\alpha_{-1}^{(s)}=0, \quad \alpha_0^{(s)}=1, \quad \nu=\left[\frac{s+1}{2}\right].$$

Lemma 4.2 is obtained from Lemma 4.1 by setting $\lambda = 0$, as a result of simple equivalent transformations of (4.3) - successive substitutions of expressions for $a_{s+1-2j}^{(s)}$ from the $(s + 1 - 2j)$-th equation of (4.3), $j = 1, 2, \ldots, \nu$, in the penultimate equation of this system.

3. Schemes for simultaneous computation of several polynomials with complex coefficients. We first consider the following scheme for computing a single polynomial of odd degree $n = 2k + 1$ containing n additions and $k+2=\left[\frac{n+3}{2}\right]$ multiplications, i.e. almost the minimum of arithmetical operations:

$$\left.\begin{aligned} g_2 &= z\cdot z = z^2,\\ p_1 &= z+\lambda_1,\\ p_{2i+1} &= p_{2i-1}(g_2+\lambda_{2i})+\lambda_{2i+1} \quad (i=1,\,2,\,\ldots,\,r),\\ P_{2k+1}(z) &= \sum_{l=0}^{2k+1} a_l z^{2k+1-l} \equiv a_0 p_{2k+1}. \end{aligned}\right\} \tag{4.5}$$

The identity in (4.5) denotes, as usual, that the equation is satisfied identically by all possible sets of complex values of $z, a_0, a_1, \ldots, a_n$.

LEMMA 4.3. *There exist algebraic functions*

$$\lambda_j=\lambda_j(a_0,\,a_1,\,\ldots,\,a_{2k+1}) \qquad (j=1,\,2,\,\ldots,\,2k+1),$$

which, on substitution in (4.5), satisfy all the equations of this scheme.

PROOF. We define $p_m = \sum_{l=0}^{m} a_l^{(m)} z^{m-l}$, $a_0^{(m)} = 1$, $m = 2i + 1$ $(i = 0, 1, \ldots, k)$. From the identity $P_{2k+1}(z) \equiv a_0 p_{2k+1}$ we find that $a_l^{(2k+1)} = \frac{a_l}{a_0}$ $(l = 0, 1, \ldots, 2k + 1)$. Furthermore, from Lemma 4.2, for each i from 1 to k, the equation

$$p_{2i+1}=p_{2i-1}(g_2+\lambda_{2i})+\lambda_{2i+1}$$

is satisfied if we express λ_{2i}, λ_{2i+1} and the coefficients of $p_{2i-1} = p_{2i-1}(z)$ as algebraic functions of the coefficients of $p_{2i+1} = p_{2i+1}(z)$ so that they satisfy (4.4) for $s = 2i - 1$, $\lambda' = \lambda_{2i}$, $\lambda'' = \lambda_{2i+1}$. It is easy to verify that in the given case $s = 2i - 1$ for any set of $a_j^{(2i+1)}$ $(j = 1, 2, \ldots, 2i + 1)$ the system (4.4) has a solution and, consequently, the required algebraic functions exist.

We shall now consider a scheme for the simultaneous computation of two polynomials of even degrees greater than two:

$$\left.\begin{aligned}
g_2 &= z\cdot z = z^2,\\
g_3 &= (g_2+\lambda_1)(z+\tilde{\lambda}_1),\\
p_3 &= g_3+\lambda_2,\\
p_{2i+1} &= p_{2i-1}(g_2+\lambda_{2i-1})+\lambda_{2i} \quad (i=2,3,\dots,k-1),\\
p_{2k} &= p_{2k-1}(z+\lambda_{2k-1})+\lambda_{2k},\\
P_{2k}(z) &= \sum_{l=0}^{2k} a_l z^{2k-l} \equiv a_0 p_{2k},\\
\tilde{p}_3 &= g_3+\tilde{\lambda}_2,\\
\tilde{p}_{2j+1} &= \tilde{p}_{2j-1}(g_2+\tilde{\lambda}_{2j-1})+\tilde{\lambda}_{2j} \quad (j=2,3,\dots,\tilde{k}-1),\\
\tilde{p}_{2\tilde{k}} &= \tilde{p}_{2\tilde{k}-1}(z+\tilde{\lambda}_{2\tilde{k}-1})+\tilde{\lambda}_{2\tilde{k}},\\
\tilde{P}_{2\tilde{k}}(z) &= \sum_{m=0}^{2\tilde{k}} \tilde{a}_m z^{2\tilde{k}-m} \equiv \tilde{a}_0 \tilde{p}_{2\tilde{k}},
\end{aligned}\right\} \tag{4.6}$$

where the identities in the scheme denote that the equations are satisfied identically by all possible sets of values of z, a_0, a_1, ..., a_{2k}, $\tilde{a}_0$, $\tilde{a}_1$, ..., $\tilde{a}_{2\tilde{k}}$.

It is not difficult to check that the scheme (4.6) contains almost the minimum of operations $\dot{\times}$ and $\pm$ in the class of schemes for simultaneous computation of two polynomials $P_{2k}(z)$ and $\tilde{P}_{2\tilde{k}}(z)$, where $k \neq 1, 2$; $\tilde{k} \neq 1$.

LEMMA 4.4. *For a suitable choice of algebraic functions*

$$\lambda_j = \lambda_j(a_0, a_1, \dots, a_{2k}, \tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_{2\tilde{k}}),$$
$$\tilde{\lambda}_l = \tilde{\lambda}_l(a_0, a_1, \dots, a_{2k}, \tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_{2\tilde{k}})$$
$$(j=1, 2, \dots, 2k;\ l=1, 2, \dots, 2\tilde{k};\ k\geqslant 3;\ \tilde{k}\geqslant 2)$$

all the equations of (4.6) *are satisfied.*[1]

PROOF. Let us first suppose that we have a fixed value for $\tilde{\lambda}_1$. We find the leading coefficients and their immediate successors in the polynomials $p_{2i+1}(z)$, $\tilde{p}_{2j+1}(z)$ ($i = 1, 2, \dots, k-1$; $j = 1, 2, \dots, \tilde{k}-1$) in scheme (4.6). They are the same for all polynomials: the leading coefficients are equal to 1 and the next after the leading coefficients are $\tilde{\lambda}_1$. Hence

---

1 If $k = 2$, $\tilde{k} = 2$, then the assertion of Lemma (4.4) is not true when simultaneously

$$\frac{a_1}{a_0} = \frac{\tilde{a}_1}{\tilde{a}_0}, \qquad \frac{a_2}{a_0} \neq \frac{\tilde{a}_2}{\tilde{a}_0}.$$

Otherwise it is true. Hence we can always compute a pair of polynomials $P_{2k}(z)$ and $\tilde{P}_{2\tilde{k}}(z)$ either by scheme (4.6) or by the scheme (4.7) which is obtained by the use of scheme (0.5) for both $P_{2k}(z)$ and $\tilde{P}_{2\tilde{k}}(z)$ i.e. with an extra addition.

A pair of polynomials $P_4(z)$, $\tilde{P}_4(z)$ whose values cannot be computed by scheme (4.6) will be called "difficult".

$$\widetilde{\lambda}_1 = \frac{a_1}{a_0} - \lambda_{2k-1} = \frac{\widetilde{a}_1}{\widetilde{a}_0} - \widetilde{\lambda}_{2\widetilde{k}-1}.$$

From the equation $p_{2k} = p_{2k-1}(z + \lambda_{2k-1}) + \lambda_{2k}$ we obtain an expression for λ_{2k} in terms of $\widetilde{\lambda}_1$ and the coefficients $a_0, a_1, \ldots, a_{2k}$ and from the equation $\widetilde{p}_{2\widetilde{k}} = \widetilde{p}_{2\widetilde{k}-1}(z + \widetilde{\lambda}_{2\widetilde{k}-1}) + \widetilde{\lambda}_{2\widetilde{k}}$ an expression for $\widetilde{\lambda}_{2\widetilde{k}}$ in terms of $\widetilde{\lambda}_1, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}}$.

By the same methods as in the proof of Lemma 4.2, we can now find algebraic functions $\lambda_i = \lambda_i^*(\widetilde{\lambda}_1, a_0, a_1, \ldots, a_{2k})$, $(j = 1, 2, \ldots, 2k - 2)$, such that after substituting these in (4.6), all the equations (4.6) from $g_2 = z^2$ to the identity $\sum_{l=0}^{2k} a_l z^{2k-l} \equiv a_0 p_{2k}$ inclusive are satisfied for arbitrary choices of the coefficients. Then in exactly the same way we find algebraic functions

$$\lambda_1 = \mu_1(\widetilde{\lambda}_1, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}}),$$
$$\widetilde{\lambda}_j = \mu_j(\widetilde{\lambda}_1, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}}) \ (j = 2, 3, \ldots, 2\widetilde{k} - 2),$$

Substituting these in (4.6) we find that all the equations in (4.6) coming after the identity $\sum_{l=0}^{2k} a_l z^{2k-l} = a_0 p_{2k}$, and also the first two equations, are satisfied for any set of coefficients $\widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}}$.

We shall prove that we can always choose

$$\widetilde{\lambda}_1 = \widetilde{\lambda}_1(a_0, \ldots, a_{2k}, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}}),$$

such that $\lambda_1^*(\widetilde{\lambda}_1, a_0, a_1, \ldots, a_{2k}) = \mu_1(\widetilde{\lambda}_1, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}})$ and, moreover, the function $\widetilde{\lambda}_1 = \widetilde{\lambda}_1(a_0, a_1, \ldots, \widetilde{a}_{2\widetilde{k}})$ is algebraic. Then, substituting $\widetilde{\lambda}_1 = \widetilde{\lambda}_1(\widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}})$ in the expressions for λ_i^* and μ_j we obtain the required set of algebraic functions of the coefficients of $P_{2k}(z)$, $\widetilde{P}_{2\widetilde{k}}(z)$ for which all the equations (4.6) are satisfied. So we establish Lemma 4.4.

The unknown algebraic function $\widetilde{\lambda}_1 = \widetilde{\lambda}_1(a_0, a_1, \ldots, a_{2k}, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}})$ can be found if and only if the function $\lambda_1^* - \mu_1$ depends on $\widetilde{\lambda}_1$, i.e. $\frac{\partial(\lambda_1^* - \mu_1)}{\partial \widetilde{\lambda}_1} \not\equiv 0$. for any set of $a_0, a_1, \ldots, a_{2k}, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}}$.

We fix the coefficients $a_0, a_1, \ldots, a_{2k}, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}}$ and we let $\widetilde{\lambda}_1$ tend to infinity together with $\lambda_{2k-1} = \frac{a_1}{a_0} - \widetilde{\lambda}_1$. We shall prove that then $\lambda_1^* - \mu_1 = \lambda_1^*(\widetilde{\lambda}_1, a_0, a_1, \ldots, a_{2k}) - \mu_1(\widetilde{\lambda}_1, \widetilde{a}_0, \widetilde{a}_1, \ldots, \widetilde{a}_{2\widetilde{k}})$ also tends to infinity, and hence we obtain the required relation $\frac{\partial(\lambda_1^*-\mu_1)}{\partial\tilde{\lambda}_1}\not\equiv 0$.

We define

$$p_{2i+1}=\sum_{j=0}^{2i+1}\alpha_j^{(2i+1)}z^{2i+1-j},\quad \alpha_0^{(2i+1)}=1\quad (i=1,\ \dots,\ k-1).$$

From Lemma 4.1 and the equations (4.2) with $s = 2k - 1$ we obtain

$$\alpha_l^{(2k-1)}=\tilde{\lambda}_1^l(1+o(1))\quad (l=0,\ 1,\ \dots,\ 2k-1).$$

Further, with the help of Lemma 4.2 and the equations (4.4) with $s = 2i - 1$, we can express, for each i from 2 to $k - 1$, the coefficients of p_{2i-1} and also λ_{2i-1}, λ_{2i} in terms of the coefficients of p_{2i+1}. Taking $i = k - 1$, $i = k - 2$, ..., $i = 2$, in succession, we obtain, in $k - 2$ steps, algebraic expressions for all λ_l ($l = 3, 4, \dots, 2k - 2$) in terms of the coefficients of p_{2k-1}. At each step, first i values of λ_{2i-1} are obtained from an algebraic equation of degree i (see the last equation in (4.4)) in the known coefficients of p_{2i+1}, then one of these values is fixed and λ_{2i} and the coefficients of p_{2i-1} are determined from (4.4) where $s = 2i - 1$, $\lambda' = \lambda_{2i-1}$, $\lambda'' = \lambda_{2i}$.

We note that the values

$$\lambda_{2l+1},\ \alpha_q^{(2i-1)}\ (q=1,\ 2,\ \dots,\ 2i-1;\ i=2,\ 3,\ \dots,\ k-2;\ l=1,\ 2,\ \dots,\ k-2)$$

so obtained satisfy the relation

$$\max_{\substack{2\leqslant q\leqslant 2i-1\leqslant 2k-1\\ 1\leqslant l\leqslant k-2}}\{\sqrt{|\lambda_{2l+1}|},\quad \sqrt[q]{|\alpha_q^{(2i-1)}|}\}\leqslant C\,|\tilde{\lambda}_1|,$$

where C is a constant.

For each $l = 2i - 1$ and $q = 1, 2, \dots, l$; $i = 2, 3, \dots, k - 1$ we select principal parts $\bar{\lambda}_l$, $\bar{a}_q^{(l)}$ from λ_l, $a_q^{(l)}$ such that[1]

$$\frac{\sqrt{\bar{\lambda}_l}}{\tilde{\lambda}_1}=\mathrm{const}^1\qquad \frac{\sqrt[q]{\bar{\alpha}_q^{(l)}}}{\tilde{\lambda}_1}=\mathrm{const}^1$$

$$\lambda_l-\bar{\lambda}_l=o(\tilde{\lambda}_1^2),\quad \alpha_q^{(l)}-\bar{\alpha}_q^{(l)}=o(\tilde{\lambda}_1^q).$$

If in the procedure for determining λ_l($l = 3, 4, \dots, 2k - 1$) described above we replace the $a_j^{(2k-1)}$ ($j = 1, 2, \dots, 2k - 1$) by their corresponding principal parts and keep the rest of the process unaltered, then instead of λ_{2i-1}, λ_{2i} and the coefficients of p_{2i-1}, $\bar{\lambda}_{2i-1}$ and $\bar{\lambda}_{2i}$ and the principal parts of the coefficients of p_{2i-1} ($i = k - 1$, $k - 2$, ..., 2) are determined at each step. Also any two distinct values of $\bar{\lambda}_{2i-1}$ differ from each other by the quantity $\eta\,.\,\tilde{\lambda}_1^2$, where η is a non-zero constant.

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[1] This constant can depend only on the values of a_0, a_1, ..., a_{2k} which we have fixed above.

We write out the expression for $a_2^{(2i-1)}$ especially (see (4.4) with $s = 2i - 1$):

$$\alpha_2^{(2i-1)} = \alpha_2^{(2i+1)} - \lambda_{2i-1} = \alpha_2^{(2k-1)} - \sum_{j=i}^{k-1} \lambda_{2j-1} =$$
$$= \alpha_2^{(2k)} - \lambda_{2k-1}\alpha_1^{(2k-1)} - \sum_{j=i}^{k-1} \lambda_{2j-1} = \widetilde{\lambda}_1^2 - \sum_{j=i}^{k-1} \lambda_{2j-1} + o(\lambda_1^2).$$

Hence we obtain

$$\lambda_1^* = \alpha_2^{(3)} = \widetilde{\lambda}_1^2 - \sum_{i=2}^{k-1} \lambda_{2i-1} + o(\widetilde{\lambda}_1^2).$$

Similarly we can obtain

$$\mu = \widetilde{\lambda}_1^2 - \sum_{j=2}^{\widetilde{k}-1} \widetilde{\lambda}_{2j-1} + o(\widetilde{\lambda}_1^2).$$

Let us assume that the values of $\widetilde{\lambda}_{2j-1}$, and their principal parts $\widetilde{\widetilde{\lambda}}_{2j-1}$ $(j = \widetilde{k} - 1, \widetilde{k} - 2, \ldots, 2)$ have already been determined and find $\overline{\lambda}_{2i-1}$ $(i = k - 1, k - 2, \ldots, 2)$.

We have to show that in the procedure for obtaining $\overline{\lambda}_{2i-1}$ $(i = k - 1, k - 2, \ldots, 2)$ their values can be chosen such that

$$\sum_{j=2}^{\widetilde{k}-1} \widetilde{\widetilde{\lambda}}_{2j-1} - \sum_{i=2}^{k-1} \overline{\lambda}_{2i-1} \neq 0, \tag{4.8}$$

and then

$$\sum_{j=2}^{\widetilde{k}-1} \widetilde{\lambda}_{2j-1} - \sum_{i=2}^{k-1} \lambda_{2i-1} = O(\widetilde{\lambda}_1^2). \tag{4.9}$$

We assume that the values of $\overline{\lambda}_{2i-1}$ and also of $\overline{\lambda}_{2i}$ and $\overline{\alpha}_q^{(2i-1)}$ have already been fixed for $q = 1, 2, \ldots, 2i - 1$; $i = k - 1, k - 2, \ldots, 3$.

The values of $\overline{\lambda}_3$ are found from the equation (see (4.4) with $s = 3$)

$$\sum_{j=0}^{2} \overline{\alpha}_{4-2j}^{(5)} (-\overline{\lambda}_3)^j = 0. \tag{4.10}$$

If this equation has two distinct roots, then at least one of them satisfies (4.8), consequently (4.9), and Lemma 4.4 has been proved. Suppose that the two roots of (4.10) coincide. Then $\overline{\lambda}_3 = \dfrac{\overline{\alpha}_2^{(5)}}{2}$. We come back to the process for finding values of $\overline{\lambda}_{2i-1}$ one step earlier and suppose that the values of $\overline{\lambda}_{2i-1}$ are determined, together with the values of $\overline{\lambda}_{2i}$, $\overline{\alpha}_q^{(2i-1)}$ for $q = 1, 2, \ldots, 2i - 1$; $i = k - 1, k - 2, \ldots, 4$.

The values of $\overline{\lambda}_5$ are found from the equation (see (4.4) with $s = 5$)

$$\sum_{j=0}^{3} \overline{\alpha}_{6-2j}^{(7)} (-\overline{\lambda}_5)^j = 0. \tag{4.11}$$

If this equation has at least two distinct roots, then the sum

$$\bar\lambda_3+\bar\lambda_5=\frac{\bar\alpha_2^{(5)}}{2}+\bar\lambda_5=\frac{\alpha_2^{(7)}-\bar\lambda_5}{2}+\bar\lambda_5=\frac{1}{2}(\bar\alpha_2^{(7)}+\bar\lambda_5)$$

can also take two distinct values of which at least one satisfies the relation (4.8) which is equivalent to the assertion of Lemma 4.4. Consequently, if the lemma is false, then all the roots of (4.11) coincide and

$$\lambda_1^*=\alpha_2^{(3)}=\tilde\lambda_1^2-\sum_{i=2}^{k-1}\lambda_{2i-1}+o(\tilde\lambda_1^2).$$

In a similar fashion we find that if the assertion of the lemma is not fulfilled, then for each m from two to $k-1$ inclusive we have

$$\sum_{i=2}^{m}\bar\lambda_{2i-1}=\frac{m-1}{m}\bar\alpha_2^{(2m+1)}=\frac{m-1}{m}(\bar\alpha_2^{(2m+3)}-\bar\lambda_{2m+1}),$$

and consequently, for $2\leqslant l\leqslant k-1$

$$\sum_{j=0}^{l}\bar\alpha_{2l-2j}^{(2l+1)}(-\bar\lambda_{2l-1})^j, \tag{4.12}$$

is a polynomial in $\bar\lambda_{2l-1}$ of the form

$$\left(-\bar\lambda_{2l-1}+\frac{\bar\alpha_2^{(2l+1)}}{l}\right)^l.$$

However, this does not hold for the polynomial (4.12) when $l=k-1\geqslant 2$, since $\bar\alpha_q^{(2k-1)}=\tilde\lambda_1^q$. Consequently the assertion of Lemma 4.4 is true, as required.

Let us now take an arbitrary set of natural numbers $n_1, n_2, \dots, n_s$. We construct a scheme for the simultaneous computation of the polynomials $P_{n_i}^i(z)=\sum_{l=0}^{n_i}a_l^{(i)}z^{n_i-l}$, $a_0^{(i)}\neq 0$, of degrees n_i, $i=1, 2, \dots, s$ with arbitrary complex coefficients.

We compute the given polynomials of degree 2 by Horner's method and polynomials of odd degree by scheme (4.5). We compute the polynomials of even degrees greater than 2 by schemes (4.6) and (4.7), splitting them up into pairs so that the number of "difficult pairs" (see the footnote on p. 130) is minimal. Finally, if an odd number of polynomials of even degrees greater than 2, is given, then one of them, $P_{n_j}^{(j)}(z)$ is left without without a pair, and we compute it by scheme (4.5) putting $P_{2k+1}^{(j)}(z)=a_0^{(j)}zp_{2k+1}+a_{n_j}^{(j)}$, where $n_j=2k+2$, $k\geqslant 1$. Naturally, we calculate $g_2=z^2$ once only here. So we obtain the following theorem.

THEOREM 4.2. *Let $n_1, n_2, \dots, n_s$, $s\geqslant 1$, be a finite set of natural numbers containing r 2's and $l+r$ even numbers, $0\leqslant r\leqslant l+r\leqslant s$. Then a scheme can be produced for the simultaneous computation of the set of polynomials $P_{n_i}^{(i)}(z)=\sum_{l=0}^{n_i}a_l^{(i)}z^{n_i-l}$ of degrees n_i, $a_0^{(i)}\neq 0$ $(i=1, 2, \dots, s)$ and with arbitrary and independent complex coefficients,* *containing* $N - s + T$ *additions and* $\left[\frac{N+r+2}{2}\right]+\gamma_l$ *multiplications, where* T *is the minimal number of "difficult pairs"* (see the footnote on p. 130),

$$N = s + \sum_{i=1}^{s} n_i, \qquad \gamma_l = \begin{cases} 0 & \text{if } l \text{ is even,} \\ 1 & \text{if } l \text{ is odd.} \end{cases}$$

Comparing the results of Theorems 4.1 and 4.2, we see that the lower bounds for the number of operations in schemes for the simultaneous computation of the values of several given polynomials can always be attained to within T additions and from $\frac{s+r}{2}$ to $\frac{s+r+4}{2}$ multiplications depending on whether l and $N + r$ are even or odd. We shall not concern ourselves here with methods of improving the schemes further.

4. Schemes for the simultaneous computation of polynomials with real coefficients. In the real case the bounds of Theorem 4.1 can be attained to within one addition and $l + 1 + \frac{s}{2}$ multiplications, where l is the number of polynomials of even degree and s the number of polynomials to be computed.

We obtain a corresponding scheme, computing all quadratic polynomials by Horner's method and the rest by a scheme of the form (3.2) (see Theorem 3.1), naturally, however, the values of $g_2 = x^2$ and $h_2 = g_2 + x$ are computed once only for all polynomials in the given set.

There are better schemes for the simultaneous computation of the values of several polynomials of low degree with real coefficients – a case, evidently, of the greatest practical interest. In particular, if all the given polynomials have degrees at most 5, then we can produce a scheme of computation, suitable for any set of real coefficients, in which the bounds of Theorem 4.1 for the number of operations are attained to within approximately $\left[\frac{l+s}{2}\right]$ multiplications, where l is the number of polynomials of degree 4 in the given set of polynomials. To save space we omit these schemes.

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References

[1] A.M. Ostrowski, On two problems in abstract algebra connected with Horner's rule, Studies presented to R. von Mises, Academic Press, New York 1954, 40-48.

[2] T.S. Motzkin, Evaluation of polynomials and evaluation of rational functions, Bull. Amer. Math. Soc. 61 (1955), 163.

[3] J. Todd, Motivations for working in Numerical Analysis, Comm. Pure Appl. Math. 8 (1955).
= Matematika, 1 (1957), 77-79.

[4] E.C. Belaga, Some problems in the computation of polynomials, Dokl. Akad. Nauk SSSR 123 (1958), 775-777.
[5] E.C. Belaga, On computing polynomials in one variable with initial conditioning of the coefficients, Problemy Kibernet. 5 (1961), 7-15.
[6] V.Ya. Pan, Computation of polynomials by schemes with initial conditioning of coefficients and a programme for automatic determination of the parameters, Zh. Vychisl. Mat. i Mat. Fiz 2 (1962), 133-140.
= U.S.S.R. Comput. Math. math. Phys. (1963), 137-146.
[7] V.Ya. Pan, Some schemes for computing polynomials with real coefficients, Dokl. Akad. Nauk SSSR 127 (1959) 266-269.
[8] V.Ya. Pan, Some schemes for computing the values of polynomials with real coefficients, Problemy Kibernet. 5 (1961), 17-29.
[9] V.Ya. Pan, On some methods of computing the values of polynomials, Problemy Kibernet. 7 (1962), 21-30.
[10] L.A. Lyusternik, O.A. Chervonenkis and A.R. Yanpol'skii, *Matematicheskii analiz. Vychislenie elementarnykh Fynktsii* (Mathematical analysis. Computation of elementary functions) Fizmatgiz, Moscow 1963.
Translation: Handbook for Computing Elementary Functions, Pergamon, Oxford 1965.
[11] V.Ya. Pan, On computing polynomials of the fifth and sixth degrees with real coefficients, Zh. Vychisl. Mat. i Mat. Fiz. 5 (1965), 116-118.